f is continuous on set D when f is cts at every member of D

ex in 2 var → with polars

Ex: Every polynomial in n-variables is cts on $\mathbb{R}^n$.

Ex: Every rational function of n-variables is cts on its domain;

rational function?

sub Ex: $\frac{x^2-y^2}{x^2+y^2}$ is cts on its domain. This means that it's continuous everywhere but (0,0)

cts, cts, so composition is cts.

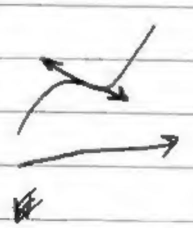
Ex: $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is cts everywhere but (0,0), at it is non-domain point.

Is it cts no matter what?

On the other hand → OTOH $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere

NB: usual rules for continuity apply (from Calc I).

Derivatives of Multivariable functions.



Idea: The derivative measures change in output from corresponding small change in input. In some direction

How do we know + or - limit of both

Defn: Let f be a function of n-variable and pick $\vec{u}$, a unit vector in $\mathbb{R}^n$. Let $\vec{a} \in \mathrm{dom}(f)$. The directional derivative of f at $\vec{a}$ in direction of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

tells how much we want to change in this direction

Ex: Compute directional derivative of $f(x,y) = xy$ at $\vec{a} = \langle 1,3\rangle$ in the direction $\vec{u} = \frac{1}{2}\langle\sqrt{2},\sqrt{2}\rangle$.

Sol: ~~Ex~~ $D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} = \lim_{h\to 0^+} \frac{f(\langle 1,3\rangle + \frac{h}{2}\langle\sqrt{2},\sqrt{2}\rangle) - f(\langle 1,3\rangle)}{h} = \lim_{h\to 0^+} \frac{f(1+\frac{\sqrt{2}h}{2}, 3+\frac{\sqrt{2}}{2}h) - f(1,3)}{h} = \lim_{h\to 0^+} \frac{(1+\frac{\sqrt{2}h}{2})\cdot(3+\frac{\sqrt{2}}{2}h) - 3}{h} =$

$= \lim_{h\to 0^+} \frac{3 + \frac{2}{4}h^2 + \frac{3\sqrt{2}}{2}h + \frac{\sqrt{2}}{2}h - 3}{h} = \lim_{h\to 0^+} \frac{\cancel{h}(\frac{h}{2}+2\sqrt{2})}{\cancel{h}} =$

$= \lim_{h\to 0^+} \frac{h}{2} + 2\sqrt{2} = 2\sqrt{2}$. ∎

$\cancel{3} + \frac{1}{2}h^2 + \frac{4\sqrt{2}}{2}h = \cancel{3} + h\left(\frac{h}{2} + 2\sqrt{2}\right)$

Exercise: Repeat the exercise with $\vec{a} = \langle x,y\rangle$.

NB: The directional derivative is very general. We want something like the "rule" from Calculus I.

Def: Let f be a function of n-variables and let $\vec{e}_k$ be the "k-th standard basis vector in $\mathbb{R}^n$", i.e. $\vec{e}_k = \langle 0,0,\dots,1,0,\dots,0\rangle$ (1 in kth position)

The k^{th} partial derivative of f (alt. partial derivative of f wrt x_k) $\quad D_{\vec{e_k}} f(\vec{a})$

1^{st} of Oct.

Last time: Derivatives of multivariate Functions

directional derivative : $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

unit vector in $\mathbb{R}^n$ — point (vector) in $\mathrm{dom}(f) \subseteq \mathbb{R}^n$

Partial derivatives $\quad x_1, x_2, \ldots, x_n$, special

vectors $\vec{e}_k = \langle \underbrace{0, \ldots,}_{0} \underset{k^{th}\text{ position}}{1}, \underbrace{\ldots, 0}_{0} \rangle$

$$\frac{\partial f}{\partial x_k} = D_{\vec{e_k}} f$$

notation for kth partial derivative

Ex. (small "what is going on?")

Let's think about $n=2$: $f(x,y)$

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = D_{\vec{e}_1} f(a,b) = \lim_{h \to 0^+} \frac{f(\langle a,b\rangle + h\vec{e}_1) - f(a,b)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(a+h\cancel{\vec{e}}, b) - f(a,b)}{h} = \lim_{h\to 0^+} \frac{f(a+h,b) - f(a,b)}{h}$$

Define $g(x)$ to be $f(x,b)$. The previous line becomes

$$\frac{\delta f}{\delta x}\Big|_{(a,b)} = \lim_{h\to 0^+} \frac{g(a+h)-g(a)}{h}$$ → the point is that the second variable is constant

← usual derivative! All the usual properties hold!

$$= g'(a)$$ (def. of derivatives) → by calc 1.

point: $\frac{\partial f}{\partial x}$ is the "usual derivative" of f, pretending that every variable except for x is constant!

Similarly, partial $\frac{\partial f}{\partial y}$ is the derivative of f, holding x constant.

Ex: Consider the partial derivatives of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$.

Sol: $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy + \sqrt{y} - \sin(x-y)\right]$$ ← use single variable derivative properties

$$= \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}\underbrace{[\sqrt{y}]}_{\text{constant wrt to } x} - \frac{\partial}{\partial x}\sin(x-y)]$$

$$= y\frac{\partial}{\partial x}[x] + 0 - \cos(x-y)\frac{\partial}{\partial x}(x-y)$$

$$= y - \cos(x-y)$$

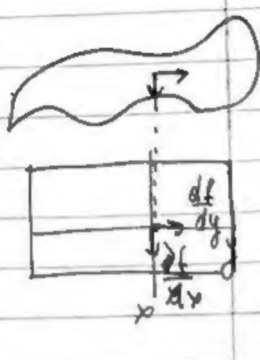
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy + \sqrt{y} - \sin(x-y)\right].$$

$$= \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial y}[\sqrt{y}] - \frac{\partial}{\partial y}[\sin(x-y)]$$

$$= x\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[\sqrt{y}] - \cos(x-y)\frac{\partial}{\partial y}[x-y]$$

$= x + \frac{1}{2\sqrt{y}} + \cos(x-y)$

Ex. Compute partial derivatives of $f(x,y,z)\, e^{x^2+y^2}\sin(xz)\cos(yz)$



$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] =$$

$$= \cos(yz)\frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\right] = \cos(yz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\sin(xz) + \right.$$

$$\left. + e^{x^2+y^2}\frac{\partial}{\partial x}\left[\sin(xz)\right]\right) = \cos(yz)\left(e^{x^2+y^2}\, 2x\sin(xz) + \right.$$

$$\left. + e^{x^2+y^2} z\cos(xz)\right) = \cos(yz)\, e^{x^2+y^2}\left(2x\sin(xz) + z\cos(xz)\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] =$$

$$= \sin(xz)\frac{\partial}{\partial y}\left[e^{x^2+y^2}\cos(yz)\right] =$$

$$= \sin(xz)\left(e^{x^2+y^2}\, 2y\cos(yz) + z(-\sin(yz)e^{x^2+y^2}\right) =$$

$$= \sin(xz)\left(2y\cos(yz)e^{x^2+y^2} - z\sin(yz)e^{x^2+y^2}\right) =$$

$$= \sin(xz)\, e^{x^2+y^2}\left(2y\cos(yz) - z\sin(yz)\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] =$$

$$e^{x^2+y^2}\frac{\partial}{\partial z}\left[\sin(xz)\cdot\cos(yz)\right] =$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) + y\sin(xz)(-\sin(yz))\right) =$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) - \underline{y\,\sin(xz)\sin(yz)}\right)$$

↓ more clear

$y\sin(xz)\sin(yz)$

NB: Higher order partial derivatives still make sense just like higher order derivatives make sense in Calc. 1

Except: There's a lot more of them

If $f(x,y)$ is given, the second order partials are:

$$\frac{\partial^2 f}{(\partial x)^2},\quad \frac{\partial^2 f}{(\partial y)^2},\quad \frac{\partial^2 f}{\partial y\,\partial x},\quad \frac{\partial^2 f}{\partial x\,\partial y}$$

2nd 1st 2nd 1st

pure partials as wrt to the same variable

mixed partial derivatives

Ex: Compute 2^{nd} order partial derivatives of $f(x,y) = xy + \sqrt{y} - \sin(x-y)$

$$\frac{\partial f}{\partial x} = y - \cos(x-y) \quad\text{and}\quad \frac{\partial f}{\partial y} = x + \frac{1}{2}y^{-1/2} + \cos(x-y)$$

Now,

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[y - \cos(x-y)\right] = 0 + \sin(x-y)\frac{\partial}{\partial x}[x-y] =$$

$$= \sin(x-y)$$

$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] =$$

$$= -\frac{1}{4}y^{-3/2} + \sin(x-y)$$

partial wrt y following x

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[y - \cos(x-y)\right] = 1 + \sin(x-y)\cdot(-1) =$$

$$= 1 - \sin(x-y)$$

partial of f wrt to x

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x + \frac{1}{2}y^{-1/2} + \cos(x-y)\right] =$$

$$= 1 + (-\sin(x-y)) = 1 - \sin(x-y)$$

look for diff equations and apple

Interlude: these are truly Calc I derivatives...
Working with 1 variable at a time allows us to do everything we were doing in Calc I.

Back to the mixed partials (somehow different!)

1) Why were these equal in our example, and can we guarantee this in future examples?

Recall some Calc I: Mean value theorem.

("nice average" value theorem)

(MVT)

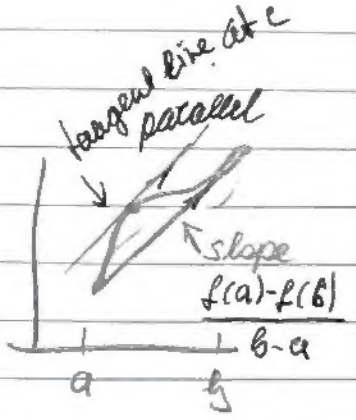
Prop: (Mean Value Theorem): Let $f(x)$ be a function that is differentiable on (a,b) and continuous on $[a,b]$. Then

$\exists c \in (a,b)$ s.t. $f'(c)(b-a) = f(b) - f(a)$

(There is $a < c < b$)

$$\left(f'(c) = \frac{f(b)-f(a)}{b-a} \right)$$

Idea: there is a point c in dom(f) or $\in (a,b)$ so that



Next time: We use MVT to prove the following:

Prop (Clairaut's theorem): Suppose $f(x,y)$ has continuous second order partial derivatives on a disk, including point (a,b). Then the second order partial derivative

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}$$